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PAPERS READ BEFORE THE ACADEMY.

I.

APPLICATIONS OF FRESNEL'S FORMULA FOR THE REFLECTION OF LIGHT.

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Read, Oct. 14, 1873.

Part I. Theoretical.

One of the most beautiful applications of the Undulatory Theory was made by Fresnel, in deducing a formula for computing the amount of light reflected by the surface of a transparent medium. He showed that if the light was polarized in the plane of incidence, that the amount reflected would be $A = \frac{\sin^2 (i - r)}{\sin^2 (i + r)}$, while, if polarized in a plane perpendicular to it, the proportion would be $B = \frac{\text{tang}^2 (i - r)}{\text{tang}^2 (i + r)}$, i and r representing the angles of incidence and refraction respectively. Natural light may be regarded as composed of two equal beams polarized at right angles, hence the amount reflected $R = \frac{1}{2} (A + B)$ $= \frac{1}{2} \left(\frac{\sin^2 (i - r)}{\sin^2 (i + r)} + \frac{\text{tang}^2 (i - r)}{\text{tang}^2 (i + r)}\right)$, a formula which may be applied to any special case, by substituting proper values for i and r. The value of A evidently increases as i varies from 0° to 90°. That of B, on the other hand, diminishes from 0° until $i + r = 90°$, when it equals 0, or at this angle, which is that of total polarization, all of the ray B is transmitted, all the reflected beam being polarized in the plane of incidence. When $i = 90°$, $A = 1$, $B = 1$, hence all the light is reflected.

When the light, instead of passing from the rare to the dense medium, goes in the other direction, the same amount of light is reflected, as we merely make $n' = \frac{1}{n}$, and i becomes r, and r, i, the values of A and B being unchanged.

When $i = 0°$, or is infinitesimal, $A = \frac{\sin^2 (n-1)r}{\sin^2 (n+1)r} = \left(\frac{n-1}{n+1}\right)^2$ and $B = \left(\frac{n-1}{n+1}\right)^2$; hence the reflected ray $R = \left(\frac{n-1}{n+1}\right)^2$ and is unpolarized. Table I. gives the values of R for various values of n.

TABLE I.

Light reflected when $i = 0°$, or incident light normal to surface.

n	$\frac{1}{2}(A+B)$	n	$\frac{1}{2}(A+B)$
1.00	0.00	2.0	11.11
1.02	0.01	2.25	14.06
1.05	0.06	2.5	18.37
1.1	0.23	2.75	22.89
1.2	0.83	3.	25.00
1.3	1.70	4.	36.00
1.4	2.78	5.	44.44
1.5	4.00	5.83	50.00
1.6	5.33	10.	66.67
1.7	6.72	100.	96.08
1.8	8.16	∞	100.00
1.9	9.63		

The light therefore increases with n, being zero when $n = 1$, and 100 per cent, or all reflected when $r = \infty$.

This law also holds for other angles of incidence, and serves to explain many familiar phenomena. The brilliancy of the diamond is mainly due to its high index, causing it to reflect about 20 per cent of the light falling normally on it, while glass returns but 4 per cent.

The white color of fine powders is generally due to the light reflected from the faces of the minute crystals of which they are composed. Hence, immersing them in a liquid diminishes their brightness, since the relative index being reduced, less light is reflected. This is seen when snow is immersed in water, or when we write with wet chalk on a blackboard. In the last case, the relative index $= \frac{\text{index chalk}}{\text{index water}} = \frac{1.5}{1.33} = 1.167$, and 0.6 per cent only is reflected; while, when dry, the index is 1.5, and 4 per cent, or six times as much, is sent back. For the same reason, white lead is injured as an oil paint when adulterated with sulphate of baryta, which, although as white, has an index of refraction of 1.68 instead of 1.9. The effect is to diminish the covering power, the light being transmitted instead of reflected, so that when applied as a paint more coats are needed. In the four cases, we have white lead, dry, index $= 1.9$, reflects 9.6 per cent; white lead in oil, index 1.27, reflection 1.44; baryta, dry, index 1.68, reflection 6.9; baryta in oil, index 1.12, reflection 0.32. Accordingly, when dry, either reflects enough light, while, when surrounded with oil, the baryta is nearly transparent. Baryta, or even carbonate of lime, may, however, be used as a water color, since either has a large enough index compared with air. With colored paints, on the other hand, we wish to destroy the reflected light which, united with the natural color of the pigment, deadens it. Hence oil colors are more brilliant than water, and the latter brighter when wet than dry. Numerous other facts may be explained in the same way, as that varnishing increases the brilliancy of pebbles and wood; in diamond mines the rough gem is distinguished from the quartz pebbles, which it resembles, by immersing it in water; and paper is rendered transparent by oiling it.

Let us next discuss the case where n is very nearly unity, or $n = 1 + dn$. Since $\sin i = n \sin r$, $\cos i \, di = \sin r \, dn$, and $di = \frac{\sin r}{\cos i} dn = \text{tang } i \, dn$. Now $i - r = di$, and $i + r = 2i + di = 2i$ $\therefore$ $A = \left(\frac{\text{tang } i \, dn}{\sin 2i}\right)^2 = \left(\frac{dn}{2 \cos^2 i}\right)^2 = \frac{dn^2}{4} \sec^4 i = \frac{dn^2}{4}(1 + \text{tang}^2 i)^2$, and $B = \left(\frac{\text{tang } i \, dn}{\text{tang } 2i}\right)^2 = \left(\frac{dn}{2}(1 - \text{tang}^2 i)\right)^2 = \frac{dn^2}{4}(1 - \text{tang}^2 i)^2$, and the degree of polarization of the reflected beam, or $\frac{A - B}{A + B} = \frac{2 \text{ tang}^2 i}{1 + \text{tang}^4 i} = \frac{2 \sin^2 i}{2 - \sin 2i}$.

Of course the absolute amount of light reflected when dn is infinitesimal is zero, unless $i = 90°$; but as commonly we only wish to compare the relative amounts when dn is small, it is generally best to neglect the constant term $\frac{di^2}{4}$, and use $A = (1 + \text{tang}^2 i)^2 = \sec^4 i$, and $B = (1 - \text{tang}^2 i)^2 = \left(\frac{\text{tang } i}{\text{tang } 2i}\right)^2$. When di varies, the reflection is proportional to its square, or to dn^2. When $i = 0$, A, B, and R become equal to 1; hence this forms an excellent unit, with which the amount reflected at other angles of incidence may be compared.

Table II. gives the amount of light for various angles of incidence. Column 1 gives i, the angle of incidence; column 2 gives A; column 3, B; column 4, $\frac{1}{2}(A + B)$, or the amount of light reflected; and column 5, the degree of polarization.

This table is of interest, as it is applicable to all cases where the two surfaces have nearly the same index. It will be noticed that the angle of total polarization is 45°, and that as i approaches 90°, the amount of light reflected becomes very great.

TABLE II.

Light reflected when n is near unity, or equals $1 + dn$.

i	A	B	$\frac{1}{2}(A+B)$	$\frac{A-B}{A+B}$
0°	1.000	1.000	1.000	0.0
5	1.015	.985	1.000	1.5
10	1.063	.939	1.001	6.2
15	1.149	.862	1.005	14.3
20	1.282	.752	1.017	26.0
25	1.482	.612	1.047	41.5
30	1.778	.444	1.111	60.0
35	2.221	.260	1.240	79.1
40	2.904	.088	1.496	94.5
45	4.000	.000	2.000	100.0
50	5.857	.176	3.016	94.5
55	9.239	1.081	5.160	79.1
60	16.000	4.000	10.000	60.0
65	31.346	12.952	22.149	41.5
70	73.079	42.884	57.981	26.0
75	222.85	167.16	195.00	14.3
80	1099.85	971.21	1035.53	6.2
85	17330.64	16808.08	17069.36	1.5
90	∞	∞	∞	0.0

The most important application of Fresnel's formula is to the case of glass, where n somewhat exceeds 1.5. Brewster calculated such a table for $n = 1.525$, which is published in Phil. Trans. 1830, p. 143. A much more complete table is given below (Tables III., IV., and V.)

TABLE III.

Light reflected when $n = 1.55$.

i	r	A	B	dA	dB	$\frac{1}{2}(A+B)$	$\frac{A-B}{A+B}$
0°	0° 0′.0	4.65	4.65	.130	.130	4.65	0.0
5	3° 13′.4	4.70	4.61	.131	.129	4.65	1.0
10	6° 25′.9	4.84	4.47	.135	.126	4.66	4.0
15	9° 36′.7	5.09	4.24	.141	.121	4.66	9.1
20	12° 44′.8	5.45	3.92	.150	.114	4.68	16.4
25	15° 49′.3	5.95	3.50	.161	.105	4.73	25.9
30	18° 49′.1	6.64	3.00	.175	.094	4.82	37.8
35	21° 43′.1	7.55	2.40	.191	.081	4.98	51.7
40	24° 30′.0	8.77	1.75	.210	.066	5.26	66.7
45	27° 8′.5	10.38	1.08	.233	.049	5.73	81.2
50	29° 37′.1	12.54	.46	.263	.027	6.50	92.9
55	31° 54′.2	15.43	.05	.303	.007	7.74	99.3
60	33° 58′.1	19.35	.12	.342	−.013	9.73	98.8
65	35° 47′.0	24.69	1.13	.375	−.032	12.91	91.2
70	37° 19′.1	31.99	4.00	.400	−.050	18.00	77.7
75	38° 32′.9	42.00	10.38	.410	−.060	26.19	61.8
80	39° 26′.8	55.74	23.34	.370	−.069	39.54	41.0
85	39° 59′.6	74.52	49.03	.250	−.061	61.77	20.6
90	40° 10′.7	100.00	100.00	.000	−.000	100.00	0.0

Angle of total polarization = 57° 10′.3, $A = 16.99$.

TABLE IV.

Light reflected when $n = 1.55$, and i is near angle of total polarization.

i	B	dB
50°	.463	+.027
51	.359	+.023
52	.263	+.019
53	.179	+.015
54	.119	+.011
55	.053	+.007
56	.016	+.003
57	.000	—.001
58	.008	—.005
59	.037	—.009
60	.120	—.013

TABLE V.

Light reflected when $n = 1.55$, and i is near 90°.

i	A	B	dA	dB	i	A	B	dA	dB
70° 0′	31.99	4.00	.400	.050	86° 0′	79.02	56.62	.209	.055
71 0	33.21	4.94	.405	.052	86 10	79.80	57.98	.201	.053
72 0	35.24	6.03	.409	.054	86 20	80.58	59.39	.193	.052
73 0	37.38	7.28	.411	.056	86 30	81.39	60.82	.187	.050
74 0	39.64	8.72	.412	.058	86 40	82.18	62.29	.178	.049
75 0	42.00	10.38	.410	.060	86 50	82.98	63.79	.171	.047
76 0	44.49	12.31	.406	.062	87 0	83.80	65.32	.163	.046
77 0	47.08	14.54	.400	.064	87 10	84.63	66.89	.155	.045
78 0	49.80	17.07	.392	.066	87 20	85.46	68.49	.147	.043
79 0	52.66	20.00	.383	.068	87 30	86.31	70.14	.140	.041
80 0	55.74	23.34	.370	.069	87 40	87.15	71.82	.133	.040
80 30	57.36	25.20	.361	.069	87 50	88.01	73.55	.125	.038
81 0	59.04	27.18	.353	.069	88 0	88.88	75.31	.118	.036
81 30	60.77	29.33	.342	.068	88 10	89.76	77.11	.109	.034
82 0	62.60	31.57	.331	.068	88 20	90.64	78.96	.099	.032
82 30	64.41	34.04	.320	.067	88 30	91.54	80.75	.090	.030
83 0	66.30	36.64	.310	.067	88 40	92.44	82.79	.081	.027
83 30	68.27	39.43	.297	.066	88 50	93.35	84.76	.072	.025
84 0	70.29	42.41	.284	.065	89 0	94.28	86.79	.063	.022
84 30	72.37	45.61	.267	.063	89 10	95.21	88.88	.054	.019
85 0	74.52	49.03	.250	.061	89 20	96.16	91.00	.045	.015
85 10	75.25	50.23	.244	.060	89 30	97.11	93.18	.035	.012
85 20	75.99	51.45	.238	.059	89 40	98.06	95.40	.024	.008
85 30	76.74	52.69	.232	.058	89 50	99.02	97.66	.012	.004
85 40	77.49	53.96	.224	.057	90 0	100.00	100.00	.000	.000
85 50	78.25	55.28	.217	.056					

for $n = 1.55$, the letters having the same meaning as in Table II. Columns 5 and 6 furnish a means of determining A and B for any other value of n. They represent the change in these quantities, for a change of n of .01. For instance, the value of A for $n = 1.55$, $i = 25°$, is 5.95, as given in the table, and it increases .161 for every increase of .01 in n. Thus, when $n = 1.50$, $A = 5.15$. In the same way B then equals 2.98. These values are of course only approximate, and are most correct for values of n between 1.50 and 1.55. In the same way, Table IV. gives the corresponding values of B, and $d\,B$ for angles near that of total polarization, and Table V. A and B for values near 90°.

In practice we commonly have to deal with some even number of parallel surfaces, especially several plates of glass. It is therefore important to discuss this case for various values of i, and of the number of plates. When a surface reflects the fraction A of the light, the transmitted portion equals $1 - A$; and if there is no internal reflection, the second surface will reflect the same proportion, or $A\,(1 - A)$, and transmit $(1 - A)^2$. In the same way, m surfaces will transmit $(1 - A)^m$. But practically, of the light reflected by the second surface, part will be turned back by the first, so that the total transmitted ray equals $\frac{1 - A}{1 + A}$, and that reflected $\frac{2\,A}{1 + A}$. In the same way it may readily be proved that for m surfaces the transmitted ray equals $\frac{1 - A}{1 + (m - 1)\,A}$, and the reflected ray $\frac{m\,A}{1 + (m - 1)\,A}$. Table VI. gives the values of i, A, and B, the amount of light reflected, the amount polarized by reflection, and that polarized by refraction, for 1, 2, 8, and 20 surfaces. The amount refracted of course equals 100 minus that reflected. The absorbed light is here neglected, as it is comparatively small, and varies with each specimen of glass.

The following conclusions are readily drawn from an examination of the numbers in Table VI., or, better, from the curves given in Figs. 1 and 2. In all the figures accompanying this paper, ordinates represent percentages, and abscissas angles of incidence. In Fig. 1, the four highest curves represent the polarization of the beams reflected by 1, 2, 8, and 20 surfaces. The other four curves give the corresponding refracted beams. Fig. 2 gives all the curves of Table VI., relating to twenty surfaces; the five curves corresponding to A, B, the intensity of the refracted beam, and the polarization of both the reflected and refracted beams. When $i = 0$, both the reflected and refracted beams are unpolarized. With ten plates of glass about half the light is reflected, the transmitted ray being but little brighter than that reflected. With 1 or 2 surfaces the reflected beam increases as i increases; with 8 surfaces it remains nearly constant up to 50°; while with 20 surfaces a marked diminution is perceived. This very remark-

TABLE VI.—Light reflected by 1, 2, 8, and 20 Surfaces.

	1 Surface.					2 Surfaces, 1 Plate.				
i	*A*	*B*	Reflect.	Per ct. Polariz.		*A*	*B*	Reflect.	Per ct. Polariz.	
				Reflct.	Refrct.				Reflct.	Refrct.
0°	4.6	4.6	4.6	0.0	0.0	8.8	8.8	8.8	0.0	0.0
5	4.7	4.6	4.6	1.0	0.0	9.0	8.7	8.8	1.0	0.1
10	4.8	4.5	4.7	4.0	0.2	9.2	8.6	8.9	4.1	0.4
15	5.1	4.2	4.7	9.1	0.4	9.8	8.1	8.9	9.2	0.9
20	5.4	3.9	4.7	16.4	0.8	10.4	7.5	8.9	16.3	1.6
25	5.9	3.5	4.7	25.9	1.3	11.1	6.7	8.9	25.4	2.5
30	6.6	3.0	4.8	37.8	1.9	12.5	5.8	9.1	36.4	3.6
35	7.5	2.4	5.0	51.7	2.7	14.2	4.7	9.4	49.3	5.2
40	8.8	1.7	5.3	66.7	3.7	16.4	3.5	10.0	64.0	7.2
45	10.4	1.1	5.7	81.2	4.9	19.0	2.1	10.6	80.1	9.4
50	12.5	0.5	6.5	92.9	6.5	22.4	0.5	11.4	95.6	12.5
55	15.4	0.1	7.7	99.3	8.3	26.9	0.1	14.0	99.3	15.6
57	17.0	0.0	8.5	100.0	9.3	29.1	0.0	14.5	100.0	17.0
60	19.3	0.1	9.7	98.8	10.6	32.4	0.2	16.3	98.8	19.2
65	24.7	1.1	12.9	91.2	13.5	38.4	2.5	20.4	87.7	23.2
70	32.0	4.0	18.0	77.7	17.1	48.5	7.7	28.1	72.6	28.4
75	42.0	10.4	26.2	61.8	21.9	59.6	19.7	39.1	52.2	33.2
80	55.7	23.3	39.5	41.0	26.8	71.7	38.0	54.8	30.7	37.2
82	62.6	31.6	47.1	32.8	29.3	77.1	48.3	62.7	22.7	38.6
84	70.3	42.4	56.3	24.7	32.1	82.7	60.2	71.4	15.7	39.6
85	74.5	49.0	61.8	20.6	33.3	85.4	68.5	75.8	12.6	40.0
86	79.0	56.6	67.2	16.5	34.8	88.2	72.5	80.3	9.8	40.4
87	83.8	65.3	74.6	12.4	36.3	91.4	79.2	85.3	7.0	40.8
88	88.9	75.3	82.1	8.2	37.9	94.1	86.1	90.1	4.4	41.0
89	94.3	86.8	90.5	4.1	39.6	96.9	92.9	94.9	2.0	41.1
90	100.0	100.0	100.0	0.0	41.2	100.0	100.0	100.0	0.0	41.2
	8 Surfaces, 4 Plates.					20 Surfaces, 10 Plates.				
0°	28.3	28.3	28.3	0.0	0.0	49.4	49.4	49.4	0.0	0.0
5	28.6	28.0	28.3	0.9	0.3	49.7	49.1	49.4	0.6	0.6
10	29.2	27.4	28.3	3.4	1.2	50.4	48.8	49.3	2.1	2.1
15	30.2	26.3	28.3	7.4	2.7	51.5	46.9	49.2	4.7	4.6
20	31.9	24.5	28.2	13.1	5.1	53.3	44.9	49.1	8.5	8.3
25	33.9	22.4	28.2	20.3	7.9	55.8	42.2	49.0	13.9	13.4
30	36.3	19.8	28.1	29.2	11.4	58.7	38.2	48.4	21.2	19.7
35	39.5	16.5	28.0	41.1	16.0	62.2	33.0	47.6	30.9	27.9
40	43.7	12.1	27.9	56.4	21.8	65.9	26.3	46.2	42.4	36.7
45	48.3	7.4	27.9	73.3	28.1	69.8	17.1	43.4	55.9	46.6
50	53.5	3.2	28.3	88.7	35.1	74.1	8.5	41.3	79.5	56.4
55	59.4	0.5	30.0	98.3	42.0	78.4	1.0	39.7	97.5	64.5
57	62.1	0.0	31.0	100.0	45.0	80.4	0.0	40.2	100.0	67.2
60	66.3	1.7	34.0	95.0	48.9	82.8	2.4	43.6	94.5	71.0
65	72.5	7.6	40.0	81.0	54.1	86.9	18.6	52.7	64.7	72.2
70	79.0	25.0	52.0	51.9	56.3	90.4	45.5	68.0	33.0	70.0
75	85.3	48.2	66.7	27.8	55.8	93.5	69.9	81.7	11.3	64.6
80	91.0	71.1	81.0	12.3	52.6	96.2	85.9	91.0	5.6	57.1
82	93.1	79.0	86.0	8.2	50.5	97.1	90.3	93.7	3.6	54.0
84	95.0	85.5	90.2	5.2	48.6	97.9	93.7	95.8	2.2	50.9
85	95.5	88.4	92.0	3.8	47.5	98.3	95.1	96.7	1.6	49.3
86	96.8	91.3	94.0	2.6	46.2	98.7	96.4	97.5	1.2	47.7
87	97.6	93.8	95.7	1.7	45.0	99.0	97.4	98.2	0.8	46.1
88	98.4	96.1	97.2	1.1	43.8	99.4	98.4	98.9	0.5	44.5
89	99.2	98.1	98.6	0.5	42.5	99.7	99.2	99.4	0.2	42.9
90	100.0	100.0	100.0	0.0	41.2	100.0	100.0	100.0	0.0	41.2

able result may be expressed by saying that 10 plates of glass transmit more light obliquely than normally. The appearance to the eye confirms this result, but it deserves a careful photometric proof. At 57° the reflected ray is of course, in all cases, totally polarized; but at other angles the amount of polarization is greater the less the number of surfaces, instead of the contrary, as might have been anticipated.

With the refracted ray quite a different law holds. For 1 surface the polarization increases from 0° to 90°; with 2 surfaces it becomes sensibly constant near 90°; while with a larger number a distinct maximum is obtained. It is commonly supposed that the greatest effect is obtained at the angle of total polarization. But the maximum is sensibly beyond this, unless a very large number of plates is employed; and hence it seems probable that a bundle of plates, polarizing by refraction, would give the best results if set at a greater angle than 57°, as 65° or 70°. The transmitted ray, however, diminishes rapidly for large angles of incidence. A very large number of plates is required to render the polarization nearly complete, which accounts for the light always remaining when even the best polariscopes by refraction are crossed. At 90° all the refracted beams are polarized by the same amount of 41.2 per cent. Or, at grazing incidence, the amount of polarization is independent of the number of plates, one polarizing as completely as a hundred. This number 41.2 may be obtained as follows. Differentiate the value of A in terms of i and r, and make $i = 90°$, when the refracted beam will equal $1 - A = 4 \operatorname{tang} r \, di = 3.376 \, di$, since when $i = 90°$, $r = 40° \, 10'.7$. In the same way $1 - B = \frac{8}{\sin 2r} \, di = 8.115 \, di$, and applying to them the formulæ for the polarization of the refracted beam, we find it equal to 41.2.

The last portion of Table VI. was determined in part by graphical interpolation, and hence the results are less accurate than those of the other tables. It is believed, however, that the errors are much less than those of any known method of testing them experimentally, and hence give all the accuracy needed for practical purposes.

To show how far these effects are due to the internal reflection, Table VII. has been computed for the same number of surfaces, supposing that no secondary reflection takes place. The first column gives the angle of incidence, the next three the polarization of the reflected, and the last three that of the refracted beam. A comparison with Table VI. shows that while the reflected beam is affected but little, a great change takes place in the transmitted light. A simple method of expressing the difference between Tables VI. and VII. is to say that, in the former, the transmitted rays are computed by the formula $\frac{1-a}{1+(m-1)a}$, and in the latter, where the internal reflection is neglected, by the formula $(1-a)^m$.

TABLE VII.

Polarization, supposing there is no internal reflection.

	Reflected Beam.			Refracted Beam.		
i	2.	8.	20.	2.	8.	20.
0°	0.0	0.0	0.0	0.0	0.0	0.0
30	37.0	32.3	24.1	3.8	15.2	36.5
45	80.3	75.1	64.0	9.9	37.6	75.6
57	100.0	100.0	100.0	18.4	63.2	95.2
70.	74.5	54.8	28.4	33.2	88.1	99.8
80	32.2	6.3	0.0	50.0	97.6	100.0
90	0.0	0.0	0.0	70.1	99.8	100.0

PART II. EXPERIMENTAL.

To test the above conclusions, two experimental methods may be employed. First, by means of a photometer, to determine the amount of light in any given case; and, secondly, by means of a polarimeter, to determine the percentage of polarization of the reflected and refracted rays. The latter method has been employed in the following experiments. The instrument commonly used to measure the amount of polarization was invented by Arago, and is called a polarimeter. It consists of a Nicol's prism and Savart's plates, in front of which are several glass plates, free to turn, and carrying an index which moves over a graduated circle, thus showing the angle through which they have been rotated. The prism and plates form a Savart's polariscope, which gives colored bands with either light or dark centre, according as the plane of the prism is parallel or perpendicular to the plane of polarization. When the plates are so placed that the light passes through them normally, they have no effect on it; but when turned, they polarize it in a plane parallel to the axis of rotation, and by an amount dependent on the angle. Let the instrument be so set that the axis of rotation shall be perpendicular to the plane of polarization, and the plates set at zero. The bands will then be visible, the centre one being bright. As the plates are turned, the bands become fainter, until their polarization neutralizes that originally present in the beam; beyond this point the bands reappear dark-centred. The amount of polarization is thus readily determined, by turning the plates until the bands disappear, when the angle is reduced to percentages by means of a table. The difficulty of computing this table is, however, the real objection to the use of this instrument. It may be determined by the formulæ given in the first part of this paper, but it of course then fails to prove them. Moreover, no account is taken of imperfect transparency, dust on the surface, and other sources of error. An excellent way of forming this table experimentally is to view through the instrument a beam of light totally polarized. If now the plane of polarization of the beam is changed, the percentage of polarization will alter, being zero when it is inclined 45° to the axis of the plates, and wholly polarized at an angle of 0° or 90°. At any angle a, the beam may be regarded as composed of two, $\cos^2 a$ polarized vertically, and $\sin^2 a$ polarized horizontally. The percentage of polarization will therefore equal $\frac{\cos^2 a - \sin^2 a}{\cos^2 a + \sin^2 a} = \cos 2\,a$, from which the polarization corresponding to any given angle is readily determined.

The result of such a comparison is given in Table VIII. Four series of observations were taken, of which the numbers obtained are given in Table XIII. From them curves were constructed, with angles of incidence as abscissas and percentages of polarization as ordinates. A curve was next drawn, coinciding with them as nearly as possible, and its ordinates are given in Table VIII., column 3; the angles of incidence are given in the first column, and the theoretical polarization in the second column of the same table. Column 4 gives the differences, and from it we see that, while the agreement is very close between 0° and 60°, above this point a marked variation is perceptible. This deviation will be further discussed in connection with Table XIII. and Fig. 8.

TABLE VIII.

Table for Arago's Polarimeter.

i	Theoretical.	Empirical.	Difference.
0°	0.0	0.0	0.0
20	5.1	5.0	—0.1
30	11.4	13.0	1.6
40	21.8	23.5	1.7
50	35.1	37.0	1.9
55	42.0	43.0	1.0
60	48.9	49.0	0.1
65	54.1	57.5	3.4
70	56.3	63.5	7.2
75	55.8	67.0	11.2
80	52.6	72.0	19.4

To avoid the defects of the above instrument, the following arrangement has been employed. A brass tube, *A B*, Fig. 3, about a foot long, is closed at one end by a double image prism, *B;* and at the other by a rectangular aperture, *A*, of such a width that its two images, as in the Arago polariscope, shall be in contact, but not overlapping. To the prism is attached a Nicol's prism, free to turn, and carrying an index, moving over a graduated circle, which shows how far it has been rotated. The tube is then mounted, so that it can be set at any altitude or azimuth, or rotated around its own axis, and three graduated circles serve to measure these quantities. In the instrument as actually constructed (Fig. 4), the whole is supported on an upright, which terminates below in a large screw, *C*, by which it may be attached to a post or tree, when used out of doors. A tube slips over this, which carries a cross-piece forming a **T**, and through the top of this passes the end of a second **T**, through the end of which the polarimeter slides. Three of these tubes are graduated to show the azimuth and altitude of the polarimeter tube, and the amount it is turned around its own axis.

The working of the instrument is as follows. If the Nicol's prism is removed, and the light unpolarized, the two images of the aperture at the end will be equally brilliant. If now the Nicol's prism is replaced and turned, the images will vary in brightness, alternately disappearing at intervals of 90°. If the light is polarized, one image will in general be brighter than the other; but by turning the Nicol's prism, certain positions will always be found in which the two images will be equal. The percentage of polarization is then readily determined from the angle through which the prism has been turned. To determine the law which connects these two, let the plane of polarization be vertical, and the line of junction of the two images parallel to it. Then call A and B the brightness of the two images respectively, in which case the polarization $p = \frac{A - B}{A + B}$. If the prism is turned through an angle v, one image will have a brightness $A \sin^2 v$, the other $B \cos^2 v$; and if they are equal, $A \sin^2 v = B \cos^2 v$, hence $p = \frac{\cos^2 v - \sin^2 v}{\cos^2 v + \sin^2 v} = \cos 2v$. The amount of polarization is then very simply found by turning the Nicol until the images are equal, then reading the angle, doubling it, and taking the cosine. Evidently there are four positions of equality of the image; and in the following experiments all four were observed, reading to tenths of a degree, and the mean taken. To make the reduction, Table IX. is employed, in which the columns headed p give the percentage of polarization corresponding to the angle a.

TABLE IX.

Table for Polarimeter: $p = \cos 2a \times 100$.

a	p	a	p	a	p	a	p
0.0°	100.00	5.0°	98.48	10.0°	93.97	15.0°	86.60
0.1	100.00	5.1	98.42	10.1	93.85	15.1	86.43
0.2	100.00	5.2	98.36	10.2	93.73	15.2	86.25
0.3	99.99	5.3	98.29	10.3	93.61	15.3	86.07
0.4	99.99	5.4	98.23	10.4	93.48	15.4	85.90
0.5	99.98	5.5	98.16	10.5	93.36	15.5	85.72
0.6	99.98	5.6	98.10	10.6	93.23	15.6	85.54
0.7	99.97	5.7	98.03	10.7	93.11	15.7	85.35
0.8	99.96	5.8	97.96	10.8	92.98	15.8	85.17
0.9	99.95	5.9	97.89	10.9	92.85	15.9	84.99
1.0	99.94	6.0	97.81	11.0	92.72	16.0	84.80
1.1	99.93	6.1	97.74	11.1	92.59	16.1	84.62
1.2	99.91	6.2	97.67	11.2	92.45	16.2	84.43
1.3	99.90	6.3	97.59	11.3	92.32	16.3	84.24
1.4	99.88	6.4	97.51	11.4	92.19	16.4	84.06
1.5	99.86	6.5	97.44	11.5	92.05	16.5	83.87
1.6	99.84	6.6	97.36	11.6	91.91	16.6	83.68
1.7	99.82	6.7	97.28	11.7	91.77	16.7	83.48
1.8	99.80	6.8	97.20	11.8	91.64	16.8	83.29
1.9	99.78	6.9	97.11	11.9	91.50	16.9	83.10
2.0	99.76	7.0	97.03	12.0	91.35	17.0	82.90
2.1	99.73	7.1	96.94	12.1	91.21	17.1	82.71
2.2	99.70	7.2	96.86	12.2	91.07	17.2	82.51
2.3	99.68	7.3	96.77	12.3	90.92	17.3	82.31
2.4	99.65	7.4	96.68	12.4	90.78	17.4	82.11
2.5	99.62	7.5	96.59	12.5	90.63	17.5	81.91
2.6	99.59	7.6	96.50	12.6	90.48	17.6	81.71
2.7	99.56	7.7	96.41	12.7	90.33	17.7	81.51
2.8	99.53	7.8	96.32	12.8	90.18	17.8	81.31
2.9	99.49	7.9	96.23	12.9	90.03	17.9	81.11
3.0	99.45	8.0	96.13	13.0	89.88	18.0	80.90
3.1	99.42	8.1	96.03	13.1	89.73	18.1	80.70
3.2	99.38	8.2	95.93	13.2	89.57	18.2	80.49
3.3	99.34	8.3	95.83	13.3	89.41	18.3	80.28
3.4	99.30	8.4	95.73	13.4	89.26	18.4	80.07
3.5	99.25	8.5	95.63	13.5	89.10	18.5	79.86
3.6	99.22	8.6	95.53	13.6	88.94	18.6	79.65
3.7	99.17	8.7	95.42	13.7	88.78	18.7	79.44
3.8	99.12	8.8	95.32	13.8	88.62	18.8	79.23
3.9	99.07	8.9	95.21	13.9	88.46	18.9	79.02
4.0	99.03	9.0	95.11	14.0	88.29	19.0	78.80
4.1	98.98	9.1	95.00	14.1	88.13	19.1	78.59
4.2	98.93	9.2	94.89	14.2	87.96	19.2	78.37
4.3	98.88	9.3	94.78	14.3	87.80	19.3	78.15
4.4	98.82	9.4	94.66	14.4	87.63	19.4	77.93
4.5	98.77	9.5	94.55	14.5	87.46	19.5	77.71
4.6	98.72	9.6	94.44	14.6	87.29	19.6	77.49
4.7	98.66	9.7	94.32	14.7	87.12	19.7	77.27
4.8	98.60	9.8	94.21	14.8	86.95	19.8	77.05
4.9	98.54	9.9	94.09	14.9	86.78	19.9	76.83

TABLE IX., — *continued.*

Table for Polarimeter: $p = \cos 2a \times 100$.

a	*p*	*a*	*p*	*a*	*p*	*a*	*p*
20.0°	76.60	25.0°	64.28	30.0°	50.00	35.0°	34.20
20.1	76.38	25.1	64.01	30.1	49.70	35.1	33.87
20.2	76.15	25.2	63.74	30.2	49.39	35.2	33.54
20.3	75.93	25.3	63.47	30.3	49.09	35.3	33.22
20.4	75.70	25.4	63.20	30.4	48.79	35.4	32.89
20.5	75.47	25.5	62.93	30.5	48.48	35.5	32.56
20.6	75.24	25.6	62.66	30.6	48.17	35.6	32.23
20.7	75.01	25.7	62.39	30.7	47.87	35.7	31.90
20.8	74.78	25.8	62.11	30.8	47.56	35.8	31.56
20.9	74.55	25.9	61.84	30.9	47.25	35.9	31.23
21.0	74.31	26.0	61.57	31.0	46.95	36.0	30.90
21.1	74.08	26.1	61.29	31.1	46.64	36.1	30.57
21.2	73.85	26.2	61.01	31.2	46.33	36.2	30.24
21.3	73.61	26.3	60.74	31.3	46.02	36.3	29.90
21.4	73.37	26.4	60.46	31.4	45.71	36.4	29.57
21.5	73.13	26.5	60.18	31.5	45.40	36.5	29.24
21.6	72.90	26.6	59.90	31.6	45.09	36.6	28.90
21.7	72.66	26.7	59.62	31.7	44.78	36.7	28.57
21.8	72.42	26.8	59.34	31.8	44.46	36.8	28.23
21.9	72.18	26.9	59.06	31.9	44.15	36.9	27.90
22.0	71.93	27.0	58.78	32.0	43.84	37.0	27.56
22.1	71.69	27.1	58.50	32.1	43.52	37.1	27.23
22.2	71.45	27.2	58.21	32.2	43.21	37.2	26.89
22.3	71.20	27.3	57.93	32.3	42.89	37.3	26.56
22.4	70.96	27.4	57.68	32.4	42.58	37.4	26.22
22.5	70.71	27.5	57.36	32.5	42.26	37.5	25.88
22.6	70.46	27.6	57.07	32.6	41.94	37.6	25.54
22.7	70.21	27.7	56.78	32.7	41.63	37.7	25.21
22.8	70.97	27.8	56.50	32.8	41.31	37.8	24.87
22.9	69.72	27.9	56.21	32.9	40.99	37.9	24.53
23.0	69.47	28.0	55.92	33.0	40.67	38.0	24.16
23.1	69.21	28.1	55.63	33.1	40.35	38.1	23.85
23.2	68.96	28.2	55.34	33.2	40.03	38.2	23.51
23.3	68.71	28.3	55.05	33.3	39.71	38.3	23.17
23.4	68.45	28.4	54.76	33.4	39.39	38.4	22.83
23.5	68.20	28.5	54.46	33.5	39.07	38.5	22.49
23.6	67.94	28.6	54.17	33.6	38.75	38.6	22.15
23.7	67.69	28.7	53.88	33.7	38.43	38.7	21.81
23.8	67.43	28.8	53.58	33.8	38.11	38.8	21.47
23.9	67.17	28.9	53.29	33.9	37.78	38.9	21.13
24.0	66.91	29.0	52.99	34.0	37.46	39.0	20.79
24.1	66.65	29.1	52.70	34.1	37.14	39.1	20.45
24.2	66.39	29.2	52.40	34.2	36.81	39.2	20.11
24.3	66.13	29.3	52.10	34.3	36.49	39.3	19.77
24.4	65.87	29.4	51.80	34.4	36.16	39.4	19.42
24.5	65.61	29.5	51.50	34.5	35.84	39.5	19.08
24.6	65.34	29.6	51.20	34.6	35.51	39.6	18.74
24.7	65.08	29.7	50.90	34.7	35.18	39.7	18.39
24.8	64.81	29.8	50.60	34.8	34.86	39.8	18.05
24.9	64.55	29.9	50.30	34.9	34.53	39.9	17.71

TABLE IX., — *continued.*

Table for Polarimeter: $p = \cos 2a \times 100$.

a	*p*	*a*	*p*	*a*	*p*	*a*	*p*
40.0°	17.36	41.3°	12.88	42.6°	8.37	43.8°	4.19
40.1	17.02	41.4	12.53	42.7	8.02	43.9	3.84
40.2	16.68	41.5	12.19	42.8	7.67	44.0	3.49
40.3	16.33	41.6	11.84	42.9	7.32	44.1	3.14
40.4	15.99	41.7	11.49	43.0	6.98	44.2	2.79
40.5	15.64	41.8	11.15	43.1	6.63	44.3	2.44
40.6	15.30	41.9	10.80	43.2	6.28	44.4	2.09
40.7	14.95	42.0	10.45	43.3	5.93	44.5	1.74
40.8	14.61	42.1	10.11	43.4	5.58	44.6	1.40
40.9	14.26	42.2	9.76	43.5	5.23	44.7	1.05
41.0	13.92	42.3	9.41	43.6	4.88	44.8	0.70
41.1	13.57	42.4	9.06	43.7	4.54	44.9	0.35
41.2	13.23	42.5	8.72				

Evidently when the light is unpolarized, the angle will be 45°; when totally polarized, 0°. We must now determine the effect when the line of junction is not parallel to the plane of polarization, but inclined at an angle w. The two images will in this case have a brightness $(A \cos^2 w + B \sin^2 w)$, and $(A \sin^2 w + B \cos^2 w)$. Hence the apparent polarization $p' = \frac{(A \cos^2 w + B \sin^2 w) - (A \sin^2 w + B \cos^2 w)}{(A \cos^2 w + B \sin^2 w) + (A \sin^2 w + B \cos^2 w)}$ $= \frac{A - B}{A + B} \cos 2w = p \cos 2w$. Hence, if the line of junction is not parallel to the plane of polarization, the observations must be reduced by dividing by $\cos 2w$. Evidently if $w = 45°$, the light appears to be unpolarized. The above discussion suggests a means of determining the direction of the plane of polarization. Make two observations of the amount of polarization, turning the polarimeter 45°. Then call p, p', p'', the true and the observed polarization in the two cases, and w the unknown angle between the line of junction in its first position and the plane of polarization. Having given p' and p'', we wish to determine p and w. Evidently $p' = p \cos 2w$, and $p'' = p \cos 2(45° - w) = p \sin 2w$. Taking their quotient gives $\tan 2w = \frac{p''}{p'}$, and the sum of their squares gives $p = \sqrt{p'^2 + p''^2}$. This method, though elegant theoretically, does not appear very accurate practically, as the plane is more accurately determined by covering the end of the polarimeter with a cap containing a plate of selenite, thus converting it into an Arago's polariscope. Then turn the tube until the two images have precisely the same color, when their line of junction will be inclined

45° to the plane of polarization. The plane may also be determined, more easily but less accurately, by removing the Nicol's prism, and turning the tube until the two images are equally bright, and adding 45° to the reading.

We next wish to determine the delicacy of this instrument in different parts of its scale. If the Nicol's prism is set at an angle v', differing slightly from its true value v, the brightness of the two images will be $A \sin^2 v'$ and $B \cos^2 v'$ respectively. Now it is commonly assumed that the difference in two such images will be perceptible, when the difference in brightness, divided by the brightness of either, equals a certain fraction $\frac{1}{a}$, in which a equals about 80. Now $\frac{1}{a} = \frac{A \sin^2 v' - B \cos^2 v'}{A \sin^2 v'} = \frac{\sin^2 v' \cos^3 v - \cos^2 v' \sin^2 v}{\sin^2 v' \cos^2 v} = \frac{4 \sin (v + v') \sin (v' - v)}{\sin^2 2 v}$; hence, since v is substantially equal to v', $\frac{1}{a} = \frac{4 \sin (v' - v)}{\sin 2 v} = \frac{4\, dv}{\sin 2 v}$; again, since $p = \cos 2 v$, $dp = -2 \sin 2 v\, dv$, and $dp = \frac{1 - p^2}{2 a} = \frac{\sin^2 2 v}{2 a}$, from which the error in the result for any unobserved difference in brightness of the two images is readily determined.

If $p = 0$, $dp = \frac{1}{2 a}$, its greatest value, which diminishes as the polarization increases, becoming zero when $p = 0$. Hence the greater the polarization, the more accurately it can be measured. If $a = 80$, $dp = \frac{1}{160}$ for its greatest value; hence the instrument should always give results within two-thirds of 1 per cent. Observation, however, shows that the error is much greater, a difference in brightness of $\frac{1}{80}$ being by no means perceptible. To determine this point, ten observations of a bright unpolarized cloud were taken, and gave a probable error of 1°.1, which corresponds to 3.8 per cent. The beam reflected from a plate of glass was then observed in the same way, and the probable error was found to be 0°.7, equal 1.0 per cent. As the polarization in this case was 87 per cent, the probable error should have been 3.8 $(1 - p^2) = 3\ 8 \times 2.43 = 0.9$, a result agreeing very closely with the observed amount. As might be expected, the error varies also with the intensity of the light, so that a sheet of unpolarized paper gave a probable error of 5.2 per cent. To compare the absolute brightness in this case with that of the cloud, the polarimeter was directed towards the latter, and its aperture half covered by the paper. The prism was then turned until the dark image of the sky just equalled the bright image of the paper. The mean angle was then found to be 25°, whence the relative brightness of the two images was found to be tang^2 25° $=$ .21. As each of the following observations is the mean of four, the probable error is reduced one-half.

The first series of observations were made on the light of the sky. The instrument was screwed into a post and levelled, the altitude and azimuth of the sun taken, and the instrument then directed towards the points to be observed. Most of these were in the same vertical plane with the sun, so that it was only necessary to determine their altitudes. The line of junction was then brought parallel to the plane of polarization; that is, turned until it was vertical, since it then lay in the plane passing through the sun. The four positions of the Nicol's prism, in which the two images were equally bright, were then observed, reading the angles to tenths of a degree, and taking the mean. The percentage of polarization was finally obtained from Table IX. In Table X., series 1 to 9 were taken at Waterville, N. H., in a valley at a height of about 1500 feet, surrounded by mountains about 4000 feet high. The air there may be regarded as very pure. Series 10 and 11 were taken upon the top of the building of the Institute of Technology, Boston. The first column in each case gives the altitude of the point observed, the second its distance from the sun, and the third the polarization corresponding to the mean of the four observations. It soon became evident that the polarization depended on the solar distance of the point under observation, and not on the altitude. This is more evident from Fig. 5, in which abscissas give the solar distances and ordinates the polarization of the above points.

TABLE X.

Sky Polarization.

Altitude.	Sun's Dist.	Polarization.
1. July 7. 6.45 A.M.		
—15	142	19.6
—30	126	38.9
—45	110	65.4
—60	95	64.9
2. July 7. 9.30 A.M.		
—20	112	66.2
—35	95	79.9
—50	80	68.1
—65	64	47.3
70	18	2.8
40	—13	2.1
25	—29	11.2
15	—40	19.3
90	35	16.9
—57	60	44.9
—47	70	64.5
—37	80	72.4
—38	88	76.9
—19	97	77.7
— 5	110	62.2
3. July 8. 7.05 A.M.		
—10	145	8.0
—20	135	16.4
—30	125	37.8
—40	115	50.9
4. July 10. 6.30 A.M.		
—15	149	18.4
—20	144	14.2
—30	133	28.6
—40	122	41.3
—50	111	55.0
—60	100	69.2
—70	90	73.8

TABLE X., — *continued.*

Sky Polarization.

5. July 15. 6.40 P.M.

Altitude.	Sun's Dist.	Polarization.
30	22	4.2
40	32	13.2
50	42	20.5
60	53	31.9
70	63	46.0
80	73	65.0
90	84	69.2
—80	94	70.9
—70	104	62.7
—60	115	53.0
—50	125	33.6
—40	135	20.1
—30	146	8.5
—20	156	2.8
—12	165	—3.5

6. July 16. 6.30 P.M.

Altitude.	Sun's Dist.	Polarization.
20	12	1.4
30	22	1.7
40	33	10.8
50	44	16.7

7. July 20. 12.20 P.M.

Altitude.	Sun's Dist.	Polarization.
60	— 8	7.0
50	—18	—1.0
40	—27	5.6
30	—36	11.5

8. July 30. 4.00 P.M.

Altitude.	Sun's Dist.	Polarization.
15	—21	—1.0
20	—15	—2.4
25	— 9	—2.8
40	7	2.4
45	12	—2.4
50	17	2.1
60	28	6.6
—60	88	72.7
—50	98	69.7
—40	108	58.2
—30	119	48.2
—20	129	29.2
—10	139	21.5

9. July 30. 6.30 P.M.

Altitude.	Sun's Dist.	Polarization.
13	3	—2.8
18	8	—1.7
28	19	—0.3
38	29	2.1
48	40	11.8
58	50	25.9
—60	113	55.6
—50	123	41.3
—40	134	21.5
—30	144	14.6
—20	155	2.4
—10	165	—1.7
—20	156	—9.8
—30	146	—2.8
—40	137	6.3

10. Sept. 12. 9.15 A.M.

Altitude.	Sun's Dist.	Polarization.
5	—40	12.2
15	—30	9.8
25	—20	5.2
66	20	3.1
76	30	10.8
86	40	10.8
—83	50	22.8
—73	60	33.9
—63	70	48.8
—52	80	57.1
—47	85	59.3
—42	90	62.7
—36	95	62.9
—31	100	57.7
—26	105	56.5
—20	110	55.0
—15	115	49.4
—10	120	43.5
— 5	125	37.1
— 0	130	31.2

11. Sept. 12. 6.00 P.M.

Altitude.	Sun's Dist.	Polarization.
— 8	170	2.8
—19	160	7.7
—29	150	12.2
—40	140	23.8
—50	130	37.1
—61	120	51.5
89	90	79.0
58	60	47.2
48	50	29.9
37	40	19.1
27	30	10.1
16	20	5.6
6	10	1.7

Before discussing these observations further, it seemed desirable to determine the polarization of other parts of the atmosphere not lying in the same vertical plane with the sun. Moreover, as the polarization of points at equal distances from the sun should be compared, the polarimeter was so mounted that its principal axis would pass through the sun. The two graduated circles would then give solar distances, and the angle from the vertical plane through the sun, instead of altitudes and azimuths. The second of these angles will be called the meridian distance, and will be regarded as positive to the right, and negative to the left of the sun. Of course, the direction of the axis should continually change, so as to follow the sun; but as great accuracy in the determination of the angles was not needed, it was found sufficient to readjust it every few minutes. Another advantage of this arrangement was, that the line of junction, being turned parallel to the axis, would always lie in the plane passing through the sun, and hence be parallel to the plane of polarization. Table XI. gives the result of five series of observations made in this way: —

TABLE XI.

Sky Polarization. Points equidistant from Sun.

M. D.	Polar.
12. July 15.	
Sun's Dist. 90°, Alt. 5°.	
75	78.6
60	74.3
45	79.8
30	77.0
0	77.2
30	74.8
45	75.2
60	76.8
75	76.8
Mean	76.7
13. July 20.	
Sun's Dist. 60°, Alt. 20°.	
0	44.1
30	41.3
60	40.7
—90	42.9
—60	45.7
—30	33.5
Mean	42.9
14. July 30.	
Sun's Dist. 90°, Alt. 48°.	
0	61.8
—30	69.0
—75	69.7
75	60.5
60	60.2
0	61.6
—55	68.7
—60	71.7
Mean	65.4
Sun's Dist. 60°, Alt. 45°.	
0	45.7
30	42.6
60	41.3
90	39.4
110	36.2
—30	49.1
—60	38.7
—90	39.7
—110	41.3
0	42.9
Mean	41.7
Sun's Dist. 120°, Alt 42°.	
0	45.7
30	39.1
Mean	42.4
Sun's Dist. 30°, Alt. 40°.	
0	13.6
60	5.2
90	7.3
120	3.5
180	5.9
—120	19.1
—90	15.3
—60	16.0
Mean	10.7
15. Sept. 12.	
Sun's Dist. 90°, Alt. 40°.	
0	66.9
30	67.9
60	66.4
75	61.6
0	66.4
30	68.2
60	65.1
75	65.6
90	54.8
0	65.6
Mean	60.4
Sun's Dist. 120°, Alt. 41°.	
0	42.9
30	44.8
45	44.5
55	41.6
0	40.5
—30	41.8
—45	37.5
—55	35.8
0	38.1
Mean	40.9
Sun's Dist. 60°, Alt. 43°.	
0	30.9
30	35.5
60	37.5
90	33.9
120	33.2
—30	35.8
—60	31.9
—90	32.6
—120	38.4
0	31.9
Mean	34.2
16. Sept. 12.	
Sun's Dist. 90°, Alt. —5°.	
0	77.9
—30	79.4
—60	79.0
—75	77.8
0	75.0
30	75.7
Mean	77.5

The first column gives in each case the meridian distance, the second the mean of the four positions of equality of the Nicol's prism, and the third the corresponding polarization. All these observations point to one very remarkable result; namely, that the polarization is the same for a given solar distance for any meridian distance; in other words, that the polarization is the same for all points equally distant from the sun. The variations in the observations are to be ascribed partly to errors of observation and partly to real irregularities in the atmosphere, as it is evident that they follow no regular law. The means therefore give us the true polarization with much greater accuracy. They are represented in Fig. 5 by small crosses. The next thing is to determine the law which connects the polarization with the solar distance in all these observations. A drawing was made like Fig. 5 enlarged, and a fine copper wire laid on it, and bent into such a shape that it should coincide as nearly as possible with all the observations. The ordinates for every 10° were then read off, giving the results entered in column 2 of Table XII.

TABLE XII.

Theoretical Formula for Sky Polarization.

S. D.	Obs.	Theor.	Differ.
0	0.0	0.0	0.0
10	1.0	1.0	0.0
20	3.5	4.2	+ 0.7
30	9.0	10.0	+ 1.0
40	17.5	18.2	+ 0.7
50	28.5	29.0	+ 0.5
60	41.0	42.0	+ 1.0
70	56.0	55.4	— 0.6
80	67.0	66.1	— 0.9
90	72.0	70.0	— 2.0
100	68.0	66.1	— 1.9
110	58.0	55.4	— 2.6
120	44.0	42.0	— 2.0
130	30.0	29.0	— 1.0
140	18.0	18.2	+ 0.2
150	8.5	10.0	+ 1.5
160	3.0	4.2	+ 1.2
170	1.0	1.0	0.0
180	0.0	0.0	0.0

A simple explanation of the polarization of the sky is to assume that it consists of molecules of air or aqueous vapor, which reflect the light specularly, and whose index of refraction differs only by a very minute amount from that of the medium in which they float. The theoretical polarization would then be at once given by Table II., making i equal to one-half the solar distance. The curve thus obtained is given in Fig. 5, at A. The polarization according to this should be complete at 90° from the sun, while in reality it is only about 70 per cent. If, however, we multiply the ordinates of curve A by this fraction, we obtain curve B, which agrees almost precisely with the curve given in column 2 of Table XII. Its ordinates are given in column 3, and the differences in column 4. From the latter it will be seen that the empirical curve gives results somewhat too great for solar distances less than 60°, and too small for greater distances; but the deviation is so small, compared with the accidental errors, that we are scarcely justified in drawing any conclusions from them. The agreement of all the observations in the neighborhood of 120° from the sun is remarkable, and not easily explained. The observations of series 10 for distances less than 110° give results decidedly below that given by theory. A possible explanation is the reflection of the sun on the sea to the east of Boston, a source of error not present in the earlier observations which were made inland. It will be noticed that no account is here taken of the points of no polarization, or neutral points of the sky; but the polarization is very slight for some distance from them, and hence is not easily measured. They must be regarded as due to some secondary disturbing cause, as refracted light, which alters the general polarization of the sky but little.

When the polarimeter is directed towards a polished colored plane surface, the two images assume different tints. One, which contains the light polarized in the plane of incidence, or B, is composed mainly of the light reflected specularly, and is therefore white like the source of light. The image A contains but little of the light reflected specularly, consisting principally of the rays emitted by the body, and hence partaking of its color. The idea at once suggested itself, that testing the light of the sky in this way might give a clue to the cause of its color. The experiment was tried several times, with negative results, the two images appearing of precisely the same blue tint. But on the evening of July 15th, near sunset, when measuring the polarization of a point near the northern horizon, where the blue color was comparatively pale, a marked difference in the two images was observable. The image B was found to be of a yellowish brown, A of a grayish blue or violet tint. This observation has since been frequently repeated, and can, in fact, be made almost any clear evening near sunset. Evidently we may conclude from these colors that the true color of the sky particles is blue, a view quite in accordance with the observations of Prof. Cooke with the spectroscope, and of Prof. Tyndall on aqueous vapor in a state of formation.

Observations were next made to test the results found above, for the light reflected and transmitted by several parallel surfaces of glass. To check the results, which are given in Table XIII., two, and in some cases three, independent methods were employed. For convenience of reference, the series are numbered, as in the observations on sky polarization. The general method employed to measure the polarization of the reflected ray was to lay one or more sheets of glass on a piece of black velvet, and render them horizontal with a spirit-level. The polarimeter was then mounted a short distance from them, carefully levelled and then turned down, so that the light should be reflected from their surfaces. Its angle of depression would then equal the complement of the angle of incidence. The line of junction of the two images was then rendered vertical, and the polarization measured in the usual way. The polarization of the sky, if clear, would introduce a large error into the results, and care was therefore taken to make these observations only on cloudy days. Although it is commonly stated that no traces of polarization can be detected in the heavens when completely covered with clouds, yet it was found to be slightly polarized in a vertical plane at such times, the effect being most marked near the horizon, and probably due to reflection from terrestrial objects. To obtain a single surface of glass, a piece was blackened on one side in the flame of a candle, in the expectation that the oil in the lamp-black, having nearly the same index of refraction as the surface to which it adhered, would prevent all specular reflection from it. But the results obtained did not agree with those computed for a single surface, and a close examination showed that double reflections were given of objects in front, as with a common plate of glass; in fact, that the lamp-black acted merely like the velvet in the other cases. The two series, Nos. 20 *a* and 21 *a*, are therefore placed with the observations on two surfaces, with which they agree very well. A piece of colored glass was next used, which gave the results in the column headed 17 *a*. Series 22 *a*, 27 *a*, 35 *a*, and 36 *a* were obtained in the same manner, using 1, 4, and 10 pieces of plate glass, laid on one another so as to form a pile.

The other measurements of the polarization of the reflected beam were obtained by quite a different method. A large Babinet's goniometer, or optical circle, was employed, the slit being removed and replaced by a Nicol's prism, which was free to turn around its axis, the angle of rotation being measured by a graduated circle and index. In the eye-piece of the observing telescope, a Nicol's prism was placed, and

TABLE XIII.

Observed Polarization of 1, 2, 8, and 20 Surfaces of Glass.

ONE SURFACE.

Reflected Beam.

i	17 *a.*	18 *b.*	19 *b.*	Theor.
0°	—	—	—	0.0
10	—	—	—	4.1
20	17.0	17.4	17.0	16.4
30	38.6	38.7	39.5	37.8
40	72.7	68.4	69.2	66.7
45	79.9	83.5	85.0	81.2
50	93.6	94.5	94.8	92.9
55	97.2	99.9	99.9	99.3
60	97.6	97.7	97.8	98.8
65	87.8	88.3	88.0	91.2
70	77.5	73.8	71.7	77.7
75	58.5	58.5	52.8	61.8
80	41.0	38.7	34.9	41.0
82	—	29.6	27.2	32.8
84	—	23.5	19.9	24.7
85	20.1	18.4	13.9	20.6
86	—	15.6	13.7	16.5
87	—	12.9	9.2	12.4
88	—	5.6	4.9	8.2
89	—	—	—	4.1

TWO SURFACES, ONE PLATE.

Reflected Beam.						Refracted Beam.			
i	20 *a.*	21 *a.*	22 *a.*	23 *b.*	Th.	24 *a.*	25 *b.*	26 *b.*	Th.
0°	—	—	2.8	0.0	0.0	0.0	—0.7	1.0	0.0
10	2.5	4.5	—	—	4.1	—	2.4	1.4	0.4
20	19.7	22.1	17.0	14.9	16.3	1.4	0.0	—	1.6
30	36.0	42.3	34.9	37.8	36.4	4.9	4.9	—	3.6
40	62.5	62.9	62.9	65.6	64.0	7.3	7.7	—	7.2
45	75.1	79.4	80.3	79.9	80.1	—	9.8	—	9.4
50	87.9	88.1	92.4	93.4	95.6	11.8	11.5	—	12.5
55	95.7	94.4	98.0	99.6	99.3	14.3	14.6	—	15.6
60	93.6	93.2	97.6	98.0	98.8	15.6	19.1	17.4	19.2
65	87.1	85.9	89.3	87.5	87.7	20.1	23.2	23.8	23.2
70	69.2	71.9	71.0	71.2	72.6	26.2	27.2	26.2	28.4
75	53.9	52.4	56.2	51.5	52.2	34.2	30.9	32.2	33.2
80	30.9	37.1	33.2	35.5	30.7	34.9	36.8	39.7	37.2
82	—	—	—	22.8	22.7	31.9	39.7	41.3	38.6
84	—	—	—	20.8	15.7	—	40.3	45.7	39.6
85	16.0	19.8	18.4	21.1	12.6	39.4	41.3	48.5	40.0
86	—	—	—	—	9.8	—	46.9	50.3	40.4
87	—	—	—	—	7.0	—	49.3	56.2	40.8
88	—	—	—	—	4.4	—	49.3	56.8	41.1
89	—	—	—	4.2	2.0	—	55.3	51.2	41.2

TABLE XIII., — *continued.*

Observed Polarization of 1, 2, 8, and 20 Surfaces of Glass.

EIGHT SURFACES, FOUR PLATES.

Reflected Beam.				Refracted Beam.						
i	27 *a.*	28 *b.*	Th.	29 *a.*	30 *b.*	31 *c.*	32 *c.*	33 *c.*	36 *c.*	Th.
0°	—	0.7	0.0	0.7	—0.3	0.0	1.7	0.5	1.2	0.0
10	4.9	—	3.4	1.4	0.3	—	—	—	—	1.2
20	16.0	14.3	13.1	6.6	5.2	4.2	9.8	—	—	5.1
30	31.9	30.6	29.2	12.9	12.5	12.9	—	—	—	11.4
40	62.9	56.5	56.4	19.8	23.5	23.8	23.5	—	—	21.8
45	75.7	73.4	73.3	27.9	31.2	—	—	—	—	28.1
50	88.1	88.3	88.7	33.5	37.5	37.1	36.5	—	—	35.1
55	96.8	99.0	98.3	41.3	43.5	45.7	—	—	—	42.0
60	92.8	96.5	95.0	50.0	51.8	49.7	48.4	52.4	53.0	48.9
65	75.5	74.8	81.0	55.6	58.8	58.5	—	57.5	59.9	54.1
70	52.1	44.8	51.9	61.6	66.4	64.3	64.0	63.5	62.4	56.3
75	33.2	20.8	27.8	67.7	71.4	64.0	66.9	65.6	66.1	55.8
80	19.8	8.0	12.3	64.3	74.5	62.9	77.5	72.7	70.5	52.6
82	—	—	8.2	66 4	81.5	—	—	79.0	75.9	50.5
84	—	—	5.2	64.5	91.3	—	—	—	—	48.6
85	11.8	5.9	3.8	61.6	89.7	—	—	—	—	47.5
86	—	—	2.6	54.2	—	—	—	—	—	46.2

TWENTY SURFACES, TEN PLATES.

Reflected Beam.					Refracted Beam.				
i	35 *a.*	36 *a.*	37 *b.*	Th.	38 *a.*	39 *b.*	40 *b.*	41 *c.*	Th.
0°	—	—	1.4	0.0	1.7	0.2	0.0	0.7	0.0
10	— 3.8	5.0	—	2.1	5.6	0.7	2.8	— 2.1	2.1
20	8.7	16.3	8 0	8.5	10 8	10.1	13.6	9 8	8 3
30	29.2	25.2	16.0	21.2	25.2	20.8	28.6	22.1	19.7
40	52.4	55.0	36.5	42.4	39.4	45.1	46.3	44.8	36.7
45	70.2	68.7	51.8	55.9	53.3	56.9	62.7	54.0	46 6
50	85.3	83.3	73.1	79.5	63.7	71.9	74.3	63.7	56.4
55	95.3	96.2	97.4	97.5	70.5	82.9	85.3	76 6	64.5
60	90.6	88.5	86.1	94.5	78.4	94.3	91.6	82.1	71.0
65	62.7	65.3	67.5	64.7	81.3	97.7	93.8	85.9	72.2
70	41.3	37.5	31.9	33.0	81.1	—	97.0	91.2	70.0
75	22.5	33.9	13.9	11.3	76.6	99.1	—	94.7	64.6
80	19.8	17.0	11.3	5.6	71.9	79.6	—	85.2	57.1
85	18.0	8.6	9.4	1.6	58.2	—	—	—	49.3

in front of it quartz wedges giving lines, which were bright or dark centred, according as the transmitted ray was polarized vertically or horizontally. On looking through the telescope, the field was seen to be traversed with lines, which disappeared only when the Nicol in the collimator was inclined 45° with the vertical. At any other angle, *a*, the vertical and horizontal components, were $\cos^2 a$ and $\sin^2 a$, and hence were equivalent to a beam polarized vertically by an amount $\cos 2a$. If now any object was inserted between the two telescopes polarizing the light horizontally p, the bands would disappear only when $p = \cos 2a$. Measuring the four positions of disappearance, and taking their mean, gave an accurate measure of the polarization by Table IX., using it as with the polarimeter described above. Another way of expressing the effect of this instrument is to say that the bands disappear when the Nicol is so turned that the plane of polarization shall be brought by the object under examination to an angle of 45°. The method of measuring the polarization of the reflected ray is now obvious. The pieces of glass are placed vertically on the centre plate between the two telescopes, the latter set at an angle of $2i$, and the glass turned until the light is reflected from its surface, so as to render the field bright. The Nicol is then turned until the bands disappear, and its position recorded. The angle between the telescopes is then altered so as to make i successively 20°, 30°, 40°, &c., and the observation repeated. Various adjustments must be made to eliminate constant errors, but they need not be detailed here. Series 18 *b* consists of observations thus made on a glass prism having an index of refraction of 1.517; series 19 *b* was made with colored glass; series 23 *b* was made with one sheet of plate-glass; and series 28 *b* and 37 *b* with 4 and 10 microscope slides respectively. The latter were used, as the thickness of the plate-glass was such that, when a number of plates were placed between the telescopes, a portion of the internal reflection would be lost.

To measure the polarization by refraction, two similar methods were employed. The plates were placed vertically over the centre of a graduated circle, and a piece of ground-glass was viewed through them by the polarimeter. The plates were then set at various angles, and the polarization measured in each case. All these observations were made in cloudy weather, to eliminate the effect of sky polarization. Series 24 *a*, 29 *a*, and 38 *a* were obtained in this manner. Other observations were made with the optical circle, placing the telescopes opposite each other, and recording the angles of the Nicol for various positions of the plates. The results are shown in columns 25 *b*, 26 *b*, 30 *b*, 39 *b*, and 40 *b*. Still a third method was employed, already described in connection with the Arago's polarimeter. Four series — columns 31 *c*, 32 *c*, 33 *c*, and 34 *c* — were thus measured with four slips of glass for microscopic slides, and one series, 41 *c*, with ten pieces of plate-glass.

To show more clearly which method of measurement was employed in each series, the letter *a* is attached to all the columns measured with the polarimeter, *b* to those measured on the optical circle with Babinet's wedges, and *c* to those in which the point of disappearance of Savart's bands was found.

It will be noticed that no observations are given of the polarization of a beam transmitted by one surface of glass. There seemed to be no easy method of measuring this quantity. It might be done by making a series of prisms of such angles that when the light was incident on one face at 10°, 20°, 30°, &c., the refracted ray would strike normally on the second face. The effect of the latter would then be nothing, so that the polarization would in this case be entirely due to the first surface.

We proceed now to discuss the results given in Table XIII. Examining the observations on a single surface of glass, we see that their concordance is much greater than in the observations of sky polarization, although two quite distinct methods were employed. The column headed "Theor." gives the theoretical polarization as given in Table VI. The observed polarization is somewhat too great for angles less than 57°, and too small for greater angles. The difference may be explained by the fact that the index of refraction was somewhat less than 1.55, hence the angle of total polarization less than 57°. The same results are shown in Fig. 6, in which abscissas represent angles of incidence, and ordinates polarization. In the case of the light reflected by two surfaces, Fig. 7, the agreement is also very close. The same may be said for the refracted beam for angles below 80°. It is difficult to observe the polarization at greater angles, as the light then passes so obliquely through the glass. There seems, however, to be a decided excess of the observed over the theoretical polarization. Two series only, of observations on the light reflected by four plates of glass, Fig. 8, were taken, as they seem to agree sufficiently well with each other, and with theory. The case of the light transmitted by four plates of glass has special importance from its application to the polarimeter of Arago. For although, of course, any other number of plates might be used, yet this number, since the eclipse of 1871, seems to have been more frequently employed. Six concordant series are given, obtained by three distinct methods, and all agree in showing a marked divergence from theory for angles greater than 60°, the observed being greater than the computed polarization.

The observations on twenty surfaces, as might be expected, present still greater discrepancies. As regards the reflected ray, Fig. 9, series 37 *b* agrees pretty well with theory, but gives a much less result than either 35 *a* or 36 *a*. Both of the latter were taken by the same method, and agree pretty well together, but differ from theory at 45° by 15 per cent. The variations of the refracted beam, Fig. 10, are still greater. As before, the observations taken by the same method as 39 *b* and 40 *b* agree, but 41 *c*, taken with the Savart, gives smaller results; series 38 *a*, with the polarimeter, still less; and theory, least of all. The errors most likely to occur, which would be common to all the observations on the refracted beams, are, first, stray light, or light entering the instrument without passing through the glass; secondly, light passing through the glass endwise, which might be recognized by its deep green color; and, thirdly, light reflected from the front surfaces of the plates. But all these errors would tend to diminish, instead of increase, the polarization; and hence, if eliminated, the divergence from theory would be still greater. Probably the true explanation is that internal reflection does not take place as completely as theory assumes, partly owing to the imperfect transparency of the medium, and partly to the dust and other impurities on the surface. Comparing the results with Table VII., which shows the effect when there is no internal reflection, we see that it makes but little difference for the reflected rays, the polarization being the same for three values of i, namely, 0°, 57°, and 90°. For the refracted ray, on the other hand, the variations are very great, amounting in the case of twenty surfaces, at 90° incidence, to over 50 per cent. We also see from Tables VI. and VII. that a partial absence of the internal reflection would account for all the results obtained, while neglecting it entirely, would cause a still greater divergence between theory and observation.

On account of the thickness of the bundle of ten plates of glass, a portion of the secondary reflection would be thrown a considerable distance to one side, especially when i is large, so that it might fall quite outside of the instrument, or even be cut off by the ends of the plates. This effect would be least marked with the polarimeter, next with the Savart, and most of all with the optical circle, on account of the small aperture of the telescope. But this is just the order in which the observations stand, all of them falling between the two theoretical curves. These observations with Tables VI. and VII. also show the effect to be expected from a bundle of plates when used to polarize light by refraction. If ten plates are employed, set, as is usual, at 57°, the polarization would be only 67.2 per cent if internal reflection takes place, but would be 95.2 if this is in any way excluded. We may, in passing, point out that an advantage might be expected in such a polariscope from an increase in the angle of incidence, the increased polarization probably more than making up for the loss of light and distortion induced by the increased obliquity of the incident rays.

The want of perfect transparency of the glass would also tend to increase the polarization by enfeebling the secondary reflection, and dirt or grease on the surface of the glass would produce the same effect. With eight surfaces, these disturbing causes are much less marked, except for large angles of incidence, and hence the agreement with theory much better. Beyond 60°, however, it becomes perceptible, producing the increased polarization noticed above. Even with a single plate of glass, this disturbing cause becomes perceptible, which probably accounts for the divergence for angles greater than 80°.

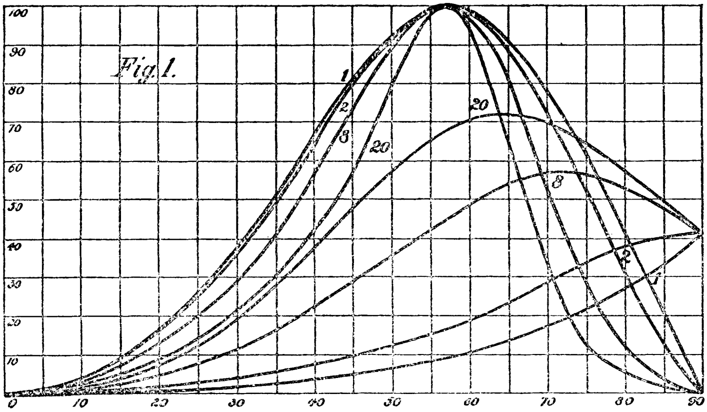
Fig. 1.
100
90
80
70
60
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90
1
2
8
20
20
8
2
1

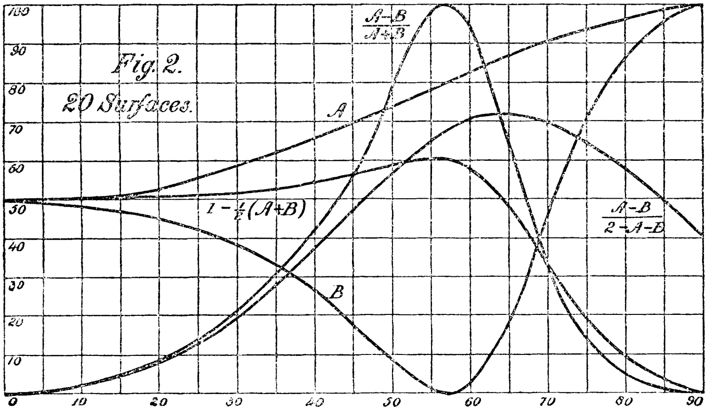
Fig. 2.
20 Surfaces.
100
90
80
70
60
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90
$\frac{A-B}{A+B}$
A
$1-\frac{1}{2}(A+B)$
B
$\frac{A-B}{2-A-B}$

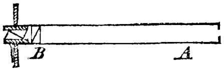
B
A

Fig. 3.

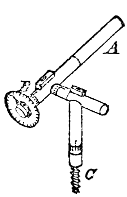
A
B
C

Fig. 4.

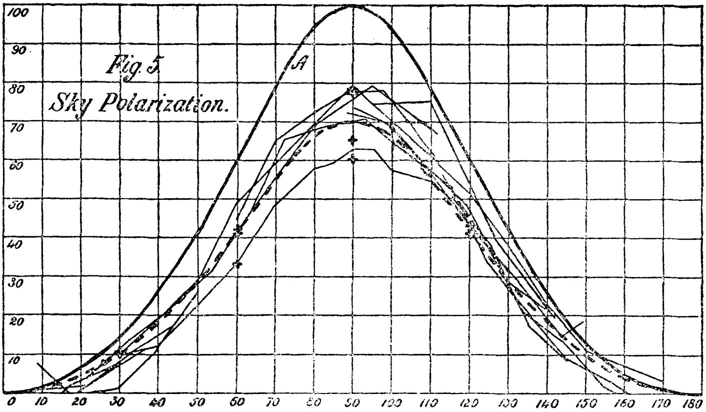

Fig. 5. Sky Polarization.

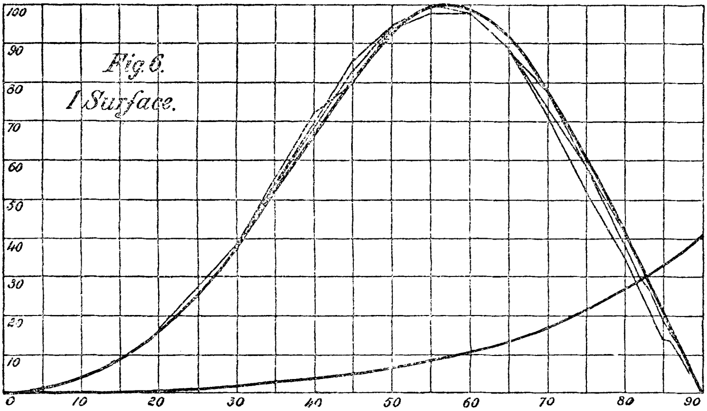

Fig. 6. 1 Surface.

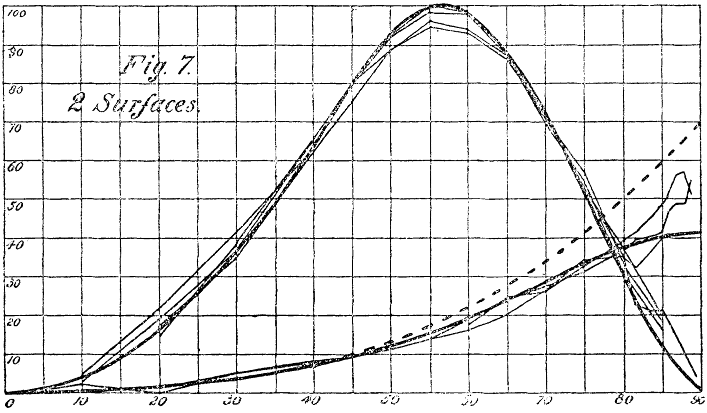

Fig. 7. 2 Surfaces.

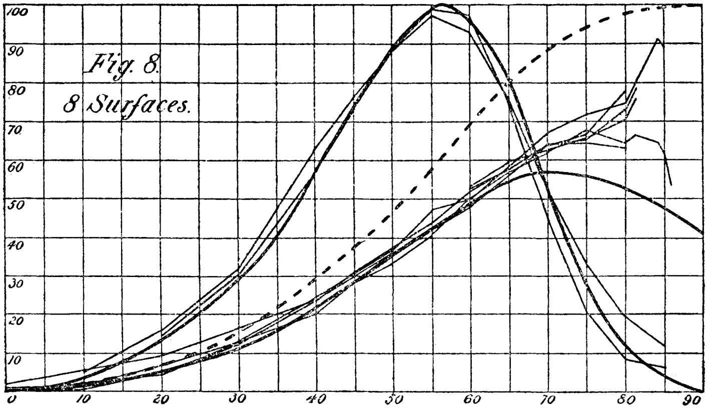

Fig. 8.
8 Surfaces.

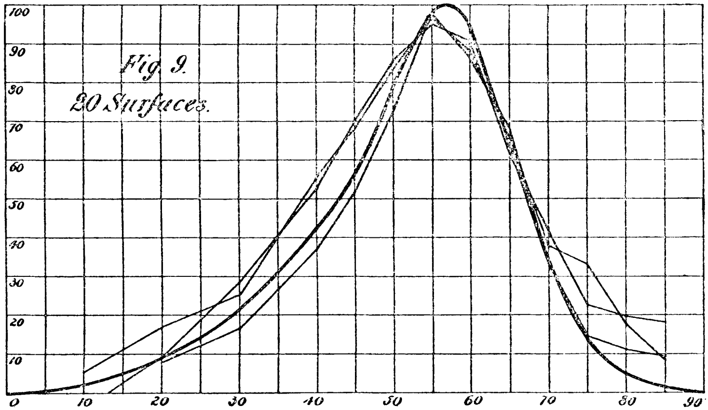

Fig. 9.
20 Surfaces.

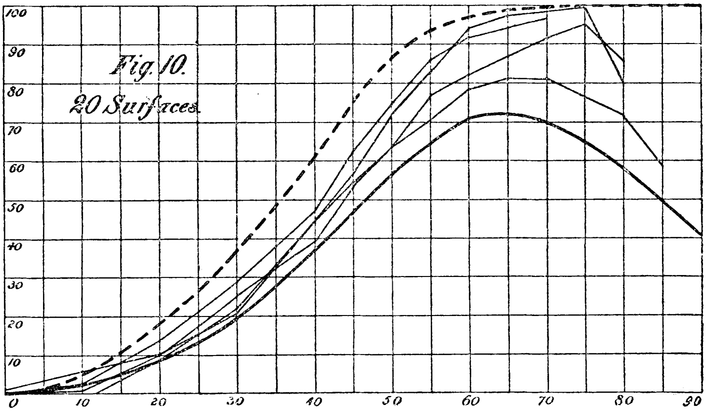

Fig. 10.
20 Surfaces.